The DTFT analysis and synthesis equations are

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$$

where $< 2\pi >$ is any contiguous interval of length $2\pi$, chosen based on whatever is most convenient for the problem at hand.

The CTFT analysis and synthesis equations are

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

This worksheet will help you explore the Fourier Transform and its properties from the two main perspectives it is used for:

1. Calculating transform pairs to find the frequency content of signals.
2. As a tool for LTI system analysis.

The mechanics are more or less the same in DT and CT. For example, the process of using Fourier Transform properties to solve for more complicated transform pairs is essentially the same for the CTFT as the DTFT, the properties are just slightly different. So, to avoid redundancy, we'll see some ways use case (1) comes up in DT through the first half of the worksheet. Then, we'll work through some examples of use case (2) in CT through the second half of the worksheet.

**Problem 1: DTFT Pairs via Brute Force** Compute the DTFT of $x[n]$ or inverse DTFT if given $X(e^{j\omega})$. You should use the analysis and synthesis equations for this question.

(a) $x[n] = \delta[n - n_0]$.

The analysis equation gives

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]e^{-j\omega n} = e^{-j\omega n_0}$$

since the delta wipes out every complex exponential except at $n = n_0$.

(b) $x[n] = \alpha^n u[n]$, where $|\alpha| < 1$.

From the analysis equation, we can obtain a convergent geometric series:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n \\ &= \frac{1}{1 - \alpha e^{-j\omega}} \end{aligned}$$

For convergence, we need

$$|\alpha e^{-j\omega}| < 1 \iff |\alpha| \cdot |e^{-j\omega}| < 1 \iff |\alpha| < 1$$

which is given as an assumption in the problem, so the series will converge. The final step comes from the fact that all complex exponentials have unit magnitude (try expanding out $|e^{-j\omega}|$ with Euler's formula to prove it if you're unconvinced).

(c) $X(e^{j\omega}) = \cos(a\omega)$, where $a$ is an integer.

First, expand the cosine as

$$X(e^{j\omega}) = \cos(a\omega) = \frac{e^{ja\omega} + e^{-ja\omega}}{2}$$

The easiest method here for determining $x[n]$ is to just pattern match this spectrum to the analysis equation. Note that the only complex exponentials present in the spectrum occur at $n = \pm a$ and each was scaled by $\frac{1}{2}$, implying the original signal must be a superposition of two deltas, each of height $\frac{1}{2}$ and at $n = \pm a$. So we must have

$$x[n] = \frac{1}{2}\left(\delta[n+a] + \delta[n-a]\right)$$

However, we can also explicitly calculate this from the synthesis equation, which will serve as a nice refresher on how to exploit the periodicity of our spectrum:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{<2\pi>} \cos(a\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{<2\pi>} \frac{e^{ja\omega} + e^{-ja\omega}}{2}e^{j\omega n}d\omega \\ &= \frac{1}{4\pi}\left[\int_{<2\pi>} e^{j\omega(n+a)}d\omega + \int_{<2\pi>} e^{j\omega(n-a)}d\omega\right] \end{aligned}$$

Recall that the integral of sine or cosine over an integer number of cycles is zero (it spends half the cycle above zero and half below, each at the same height, so the net area over a single cycle is exactly zero). So, in general, Euler's formula plus this idea tells us, for any nonzero integer $k$, that:

$$\begin{aligned} \int_{<2\pi>} e^{j\omega k} &= \int_{<2\pi>} \cos(\omega k)d\omega + j\int_{<2\pi>} \sin(\omega k)d\omega \\ &= 0 \end{aligned}$$

since the cosine and sine are both $2\pi$ periodic (they may have a smaller fundamental period, but it is easily verified that each is $2\pi$ periodic). In the special case of $k = 0$, the sine term is identically zero and the cosine identically 1, so the integral would instead be $2\pi$.

Motivated by this analysis, let's split this into three cases: $n = a, n = -a, |n| \neq a$.

For $n = a$, the first integral is zero:

$$\begin{aligned} x[a] &= \frac{1}{4\pi}\left[\underbrace{\int_{<2\pi>} e^{2aj\omega}d\omega}_{0} + \underbrace{\int_{<2\pi>} d\omega}_{2\pi}\right] \\ &= \frac{1}{2} \end{aligned}$$

For $n = -a$, the second integral is zero:

$$x[-a] = \frac{1}{4\pi}\left[\underbrace{\int_{<2\pi>} d\omega}_{2\pi} + \underbrace{\int_{<2\pi>} e^{-2aj\omega} d\omega}_{0}\right]$$
$$= \frac{1}{2}$$

Otherwise, if $|n| \neq a$, both integrals are zero. So the signal is only nonzero at $n = \pm a$, where it takes on the value $\frac{1}{2}$. Thus

$$x[n] = \frac{1}{2}\left(\delta[n+a] + \delta[n-a]\right)$$

(d) $X(e^{j\omega}) = \sin(b\omega)$, where $b$ is an integer.

We can expand the sine as

$$X(e^{j\omega}) = \sin(b\omega) = \frac{e^{jb\omega} - e^{-jb\omega}}{2j}$$

This is essentially the same as the previous part. We won't go through the step-by-step mechanics of using the synthesis equation again since there's nothing new. The pattern matching procedure leads us to the observation that the only nonzero terms in $x[n]$ occur at $n = \mp b$, which have heights $\pm 1/2j$, so

$$x[n] = \frac{1}{2j}\left(\delta[n+b] - \delta[n-b]\right)$$

If you're unsure of the signs, do a quick sanity check on this result by applying the analysis equation to verify that it yields $\sin(b\omega)$.

(e) Let $\omega_0 < \pi$. For $|\omega| \leq \pi$, the spectrum is given as $X(e^{j\omega}) = \begin{cases} 1 & \omega \in [-\omega_0, \omega_0] \\ 0 & \text{otherwise} \end{cases}$ and $2\pi$-periodically repeats outside this region.

The calculus enthusiasts will be relieved to know that the synthesis equation integral is the way to go this time. The function's limited support will reduce the integration bounds from $[-\pi, \pi]$ to $[-\omega_0, \omega_0]$:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi}\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{jn} \\
&= \frac{1}{\pi n}\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} \\
&= \frac{\sin(\omega_0 n)}{\pi n}
\end{aligned}
$$

**Problem 2: DTFT Properties for Transform Pairs** For each of the following signals or spectra, (i) identify the property or properties that allow you to reduce this to a transform pair you have already calculated on this worksheet, and, (ii) compute the DTFT if given $x[n]$ or inverse DTFT if given $X(e^{j\omega})$.

(a) $x[n] = \sum_{k=-N}^{N} \delta[n-k]$. Your answer should be of the form

$$X(e^{j\omega}) = \frac{\sin(a\omega)}{\sin(b\omega)}$$

where $a, b$ are constants you will determine. You may find the identity

$$\sum_{k=C}^{D} a^k = \frac{a^C - a^{D+1}}{1-a}$$

to be of use.

(i) The signal in question is a linear combination of signals whose Fourier Transforms we know (shifted deltas), so we should use the **linearity property**:

$$\sum_k a_k x_k[n] \overset{\mathcal{F}}{\longleftrightarrow} \sum_k a_k X_k(e^{j\omega})$$

(ii) Before doing any work, note that our time-domain signal is real and even. This means that $X(e^{j\omega})$ should also be real and even, which we can use as a nice sanity check on our final result.

Applying linearity with the transform pair

$$\delta[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}$$

derived in Q1, we obtain

$$
\begin{aligned}
X(e^{j\omega}) &= \mathcal{F}\{\sum_{k=-N}^{N} \delta[n-k]\} \\
&= \sum_{k=-N}^{N} \mathcal{F}\{\delta[n-k]\} \\
&= \sum_{k=-N}^{N} e^{-j\omega k} \\
&= \sum_{k=-N}^{N} \left(e^{-j\omega}\right)^{k} \\
&= \frac{e^{-j\omega(-N)} - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\omega(-N)} - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} \cdot \frac{e^{j\omega/2}}{e^{j\omega/2}} \\
&= \frac{e^{j\omega N + j\omega/2} - e^{-j\omega(N+1)+j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\
&= \frac{e^{j\omega(N+\frac{1}{2})} - e^{-j\omega(N+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}} \\
&= \frac{2j\sin(\omega(N+\frac{1}{2}))}{2j\sin(\omega/2)} \\
&= \frac{\sin(\omega(N+\frac{1}{2}))}{\sin(\omega/2)}
\end{aligned}
$$

So $a = N + \frac{1}{2}$ and $b = \frac{1}{2}$.
As a nice sanity check, note that our result is real and even, in agreement with the time-domain symmetry.

(b) $x[n] = \beta^n u[-n-1]$, where $|\beta| > 1$.

*Hint:* First, apply a series of transformations to get your signal into the form $\alpha^n u[n]$, where $\alpha$ is some straightforward function of $\beta$. Then, backtrack by accounting for the various transformations using Fourier Transform properties to obtain $X(e^{j\omega})$.

(i) Define $\hat{x}[n] = x[-n]$, and $\tilde{x}[n] = \hat{x}[n+1]$. Then

$$
\begin{aligned}
\tilde{x}[n] &= \hat{x}[n+1] \\
&= x[-(n+1)] \\
&= x[-n-1] \\
&= \beta^{-n-1}u[-(-n-1)-1] \\
&= \frac{1}{\beta}\left(\frac{1}{\beta}\right)^n u[n]
\end{aligned}
$$

and we know how to deal with this signal. The transformations we used involved shifting and flipping, so we'll need the **time-shift property** and the **time-reversal property**. Technically, since $\tilde{x}[n]$ is a scaled (by $\beta^{-1}$) version of a signal we know, we'll also need the **linearity property**.

(ii) Applying linearity, we know

$$
\tilde{X}(e^{j\omega}) = \frac{1}{\beta}\frac{1}{1-\frac{1}{\beta}e^{-j\omega}}
$$

Now, let's undo the transformations by one step. We defined $\tilde{x}[n] = \hat{x}[n+1]$ which by the time-shift property tells us that

$$
\tilde{X}(e^{j\omega}) = \hat{X}(e^{j\omega})e^{j\omega} \implies \hat{X}(e^{j\omega}) = \tilde{X}(e^{j\omega})e^{-j\omega}
$$

so we have

$$
\hat{X}(e^{j\omega}) = \frac{e^{-j\omega}}{\beta - e^{-j\omega}} = \frac{1}{\beta e^{j\omega}-1}
$$

Finally, since $\hat{x}[n] = x[-n]$, $X(e^{j\omega}) = \hat{X}(e^{-j\omega})$, so we determine the spectrum to be

$$
X(e^{j\omega}) = \frac{1}{\beta e^{-j\omega}-1}
$$

(c) $x[n] = \left(\frac{1}{2}\right)^{|n|}$. No need to simplify your answer.

(i) Our signal is a double-sided decaying exponential. To simplify things, we can write it as the sum of two one-sided decaying exponentials, with a correction term for $n=0$ so we don't double count:

$$
x[n] = \underbrace{\left(\tfrac{1}{2}\right)^n u[n]}_{\text{Right-sided}} + \underbrace{\left(\tfrac{1}{2}\right)^{-n} u[-n]}_{\text{Left-sided}} - \delta[n]
$$

We know how to handle the first and third terms, and the second is just the first but reversed, so we'll need the **linearity property** and the **time-reversal property**.

(ii) Define the signal $g[n]$ as $(\frac{1}{2})^n u[n]$ to simplify notation. We know from Problem 1 that

$$G(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

which we can use here. With our new notation, the signal is

$$x[n] = g[n] + g[-n] - \delta[n]$$

Applying linearity and the time-shift property, we obtain

$$\begin{aligned} X(e^{j\omega}) &= G(e^{j\omega}) + G(e^{-j\omega}) - 1 \\ &= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{1}{1 - \frac{1}{2}e^{j\omega}} - 1 \end{aligned}$$

(d) $x[n] = n\left(\frac{1}{2}\right)^{|n|}$.

(i) This is the exact same signal as in the previous part, but with an extra multiplication by $n$, so we should use the **frequency differentiation property**:

$$ny[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dY(e^{j\omega})}{d\omega}$$

(ii) From here, it's just a matter of taking the derivative of the answer to the previous question and multiplying by $j$.

$$\begin{aligned} X(e^{j\omega}) &= j\frac{dX(e^{j\omega})}{d\omega} \\ &= j\frac{d}{d\omega}\left[\frac{1}{1 - \frac{1}{2}e^{j\omega}} + \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - 1\right] \\ &= j\left(\frac{d}{d\omega}\left[\frac{1}{1 - \frac{1}{2}e^{j\omega}}\right] + \frac{d}{d\omega}\left[\frac{1}{1 - \frac{1}{2}e^{-j\omega}}\right]\right) \\ &= j\left[\frac{\frac{1}{2}je^{j\omega}}{(1 - \frac{1}{2}e^{j\omega})^2} - \frac{\frac{1}{2}je^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})^2}\right] \\ &= \frac{1}{2}\left[-\frac{1}{(1 - \frac{1}{2}e^{j\omega})^2} + \frac{1}{(1 - \frac{1}{2}e^{-j\omega})^2}\right] \end{aligned}$$

(e) $X(e^{j\omega}) = \cos(a\omega)\sin(b\omega)$, where $a, b$ are both integers. Note that this assumption on $a, b$ will guarantee a $2\pi$-periodic spectrum. **You do not need any trigonometric identities.**

(i) The spectrum is a product of spectra for which we know the corresponding time domain signals, so we should use the **convolution property**:

$$(x_1 * x_2)[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})X_2(e^{j\omega})$$

In general, explicitly computing a convolution should be thought of as a last resort, since it can quickly grow from tedious to intractable depending on the signals in question. However, since the corresponding time domain signals are sparse (each is a superposition of two deltas), it's very easy to do here and much easier than using the synthesis equation, even if we do use a trigonometric identity.

(ii) Recall the transform pairs derived in Q1:

$$\begin{aligned} x_c[n] &= \frac{1}{2}\left(\delta[n+a] + \delta[n-a]\right) \overset{\mathcal{F}}{\longleftrightarrow} X_c(e^{j\omega}) = \cos(a\omega) \\ x_s[n] &= \frac{1}{2j}\left(\delta[n+b] - \delta[n-b]\right) \overset{\mathcal{F}}{\longleftrightarrow} X_s(e^{j\omega}) = \sin(b\omega) \end{aligned}$$

Using the convolution property along with the properties of convolution (it distributes over addition, and convolving with a scaled and shifted delta just scales and shifts the corresponding signal), we find the time-domain signal as:

$$\begin{aligned} x[n] &= (x_c * x_s)[n] \\ &= \left[\frac{1}{2}\left(\delta[n+a] + \delta[n-a]\right)\right] * \left[\frac{1}{2j}\left(\delta[n+b] - \delta[n-b]\right)\right] \\ &= \frac{1}{4j}\left(\delta[n+a+b] + \delta[n+b-a] - \delta[n-b+a] - \delta[n-b-a]\right) \end{aligned}$$

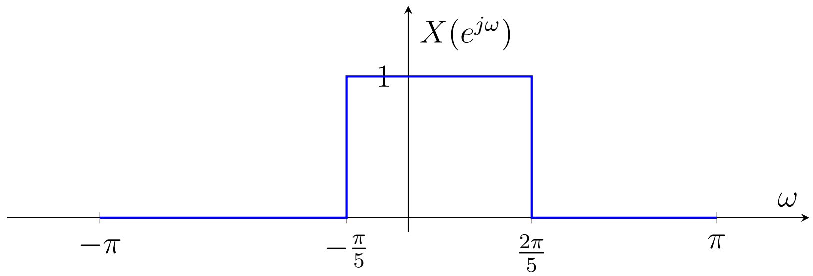
As an added bonus, note that the signal nicely separated into four deltas in the time-domain. So, if we took the Fourier Transform of this signal, we could prove the identity:

$$\cos(a\omega)\sin(b\omega) = \frac{1}{2}\left(\sin((a+b)\omega) - \sin((a-b)\omega)\right)$$

(f) The spectrum $X(e^{j\omega})$ is shown below for $\omega \in [-\pi, \pi]$, and $2\pi$-periodically repeats outside this region. **You do not need to compute any integrals.**

(i) The spectrum is a shifted version of the one we saw in Q1e, so we want to use the **shift in frequency property**:

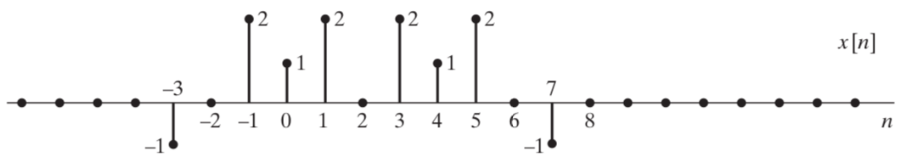
$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$




(ii) The spectrum is shifted by $\frac{\pi}{10}$ to the right of one symmetric with support on $[-3\pi/10, 3\pi/10]$. So $\omega_0 = \frac{\pi}{10}$, and we have:

$$x[n] = e^{j\frac{\pi}{10}n}\frac{\sin(\frac{3\pi}{10}n)}{\pi n}$$

**Problem 3: DTFT Properties for Salient Frequency-Domain Features** [1] Let $X(e^{j\omega})$ denote the Fourier Transform of the signal $x[n]$ shown below.




Without computing $X(e^{j\omega})$, determine the following:

(a) $X(e^{j\omega})|_{\omega=0}$.

The analysis equation tells us that

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

So, in particular,

$$X(e^{j\omega})|_{\omega=0} = \sum_{n=-\infty}^{\infty} x[n] = 8$$

This is a nice property to keep in mind: the DC component ($\omega = 0$) of a signal is just the sum of all its values.

(b) $X(e^{j\omega})|_{\omega=\pi}$.

The analysis equation tells us that

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

So, in particular,

$$X(e^{j\omega})|_{\omega=\pi} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} = \sum_{n=-\infty}^{\infty} x[n](-1)^n$$

which gives

$$X(e^{j\omega})|_{\omega=\pi} = 1 - 2 + 1 - 2 - 2 + 1 - 2 + 1 = -4$$

[1] Adapted from Problem 2.55 of *Discrete-Time Signal Processing*, 3rd edition, by Oppenheim and Schafer.

(c) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

The synthesis equation tells us that

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega n}d\omega$$

So, in particular,

$$\int_{-\infty}^{\infty} X(e^{j\omega})d\omega = 2\pi x[0] = 2\pi$$

(d) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$.

Here, we can apply Parseval's theorem:

$$\begin{aligned}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega &= 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ &= 2\pi(1+4+1+4+4+1+4+1) \\ &= 40\pi\end{aligned}$$

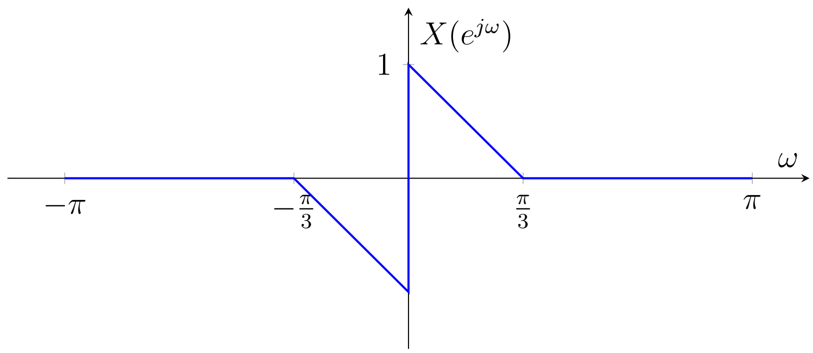
(e) Is the spectrum conjugate-symmetric, that is, does $X(e^{-j\omega})$ equal $X^*(e^{j\omega})$?

Yes, as this is generally true of any real signal.

(f) Is the spectrum even-symmetric, that is, does $X(e^{-j\omega})$ equal $X(e^{j\omega})$? If not, what transformation could we apply to the signal to make this true?

No. We have an even-symmetric spectrum if and only if we have an even-symmetric signal. However, simply shifting the signal left by two to obtain $x[n+2]$ would give us an even symmetric spectrum $X(e^{j\omega})e^{j2\omega}$.

**Problem 4: DTFT Properties for Salient Time-Domain Features** Let $x[n]$ be the signal whose Fourier Transform $X(e^{j\omega})$ is shown below over $[-\pi, \pi]$ and $2\pi$-periodically replicates outside this region.




Regarding the discontinuity, assume $X(e^{j\omega})|_{\omega=0} = 0$. Without computing $x[n]$, determine the following:

(a) $x[0]$.

We could exploit the fact that $x$ will be odd-symmetric due to the symmetry of its Fourier Transform to just say $x[0] = 0$, but we can derive this in a more principled manner as well. The synthesis equation gives:

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0$$

due the the odd symmetry of the spectrum.

(b) $\sum_{n=-\infty}^{\infty} x[n]$.

Again, this will be zero due to odd symmetry, but we can derive it from the analysis equation:

$$\sum_{n=-\infty}^{\infty} x[n] = X(e^{j\omega})|_{\omega=0} = 0$$

(c) $\sum_{n=-\infty}^{\infty} |x[n]|^2$.

We can use Parseval's theorem, and instead consider $|X(e^{j\omega})|^2$. From the graph, it's clear that this will be real and positive, and also even-symmetric. We can turn our attention to the region $\omega \in [0, \frac{\pi}{3}]$ - the region $[-\frac{\pi}{3}, 0]$ will be its mirror image.

Before integrating, we need to find the functional form of the squared spectrum. The original spectrum linearly decreases from the point $(\omega = 0, X(e^{j\omega}) = 1)$ to $(\omega = \frac{\pi}{3}, X(e^{j\omega}) = 0)$. This is a line of slope

$$\frac{\text{Rise}}{\text{Run}} = \frac{-1}{\pi/3} = -\frac{3}{\pi}$$

with vertical intercept $1$, so the spectrum can be written as

$$X(e^{j\omega}) = 1 - \frac{3}{\pi}\omega$$

for $\omega \in [0, \frac{\pi}{3}]$. So, for $\omega \in [-\pi, \pi]$, we have:

$$|X(e^{j\omega})|^2 = \begin{cases} (1 - \frac{3}{\pi}\omega)^2 & |\omega| < \frac{\pi}{3} \\ 0 & \text{otherwise} \end{cases}$$

Using this, we can find the time-domain energy:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} \left(1 - \frac{3}{\pi}\omega\right)^2 d\omega \\
&= \frac{1}{\pi}\int_{0}^{\pi/3} \left(1 - \frac{3}{\pi}\omega\right)^2 d\omega \\
&= \frac{1}{\pi}\int_{0}^{\pi/3} \left(1 - \frac{6\omega}{\pi} + \frac{9\omega^2}{\pi^2}\right) d\omega \\
&= \frac{1}{\pi}\left[\omega - \frac{3\omega^2}{\pi} + \frac{3\omega^3}{\pi^2}\right]_0^{\pi/3} \\
&= \frac{1}{9}
\end{aligned}$$

(d) An expression for $x[-n]$ in terms of $x[n]$.

$x[-n] = -x[n]$. A real and odd Fourier Transform corresponds to a purely imaginary, odd signal, and this is the definition of odd symmetry.

(e) An expression for Re $\{x[n]\}$ in terms of $x[n]$.

Re $\{x[n]\} = 0$. A real and odd Fourier Transform corresponds to a purely imaginary, odd signal. Thus, the real part is zero.

(f) An expression for Im $\{x[n]\}$ in terms of $x[n]$.

Im $\{x[n]\} = x[n]/j$. A real and odd Fourier Transform corresponds to a purely imaginary, odd signal. Thus, the imaginary part of the signal times $j$ *is* the signal, since $x[n] = \underbrace{\text{Re}\{x[n]\}}_{0} + j\text{Im}\{x[n]\}$.

**Problem 5: CTFT for System Analysis** [2] A signal $x(t)$ is the input to an LTI system $H$ with impulse response $h(t) = \frac{\sin(500\pi t)}{\pi t}$. Throughout this problem, you may find the Fourier Transform pair

$$s(t) = \frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} S(j\omega) = \begin{cases} 1 & \omega \in [-W, W] \\ 0 & \text{otherwise} \end{cases}$$

to be useful.

(a) Suppose we wanted to calculate the system's frequency response $H(j\omega)$. Without doing any math, and just inspecting $h(t)$, which of the following do we know about $H(j\omega)$? Circle all that apply.

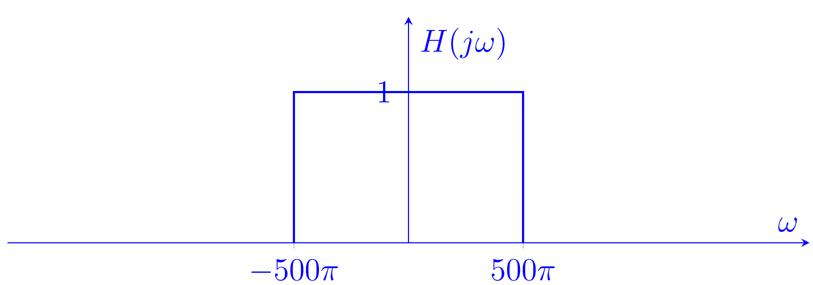
| | |
|---|---|
| It is purely real | It is purely imaginary |
| It is even-symmetric | It is odd-symmetric |

Note that $h(t)$ is real and even, so $H(j\omega)$ will be as well.

It's worth pointing out that each row contains a set of mutually exclusive properties (assuming $h(t) \neq 0$ identically, which is true here). Try proving to yourself that any function (e.g., signal, spectrum, etc.) that is both even and odd symmetric must be identically zero.

(b) Provide a well-labeled plot of the system's frequency response $H(j\omega)$.

Inspection of the transform pair given above shows that our spectrum is a unit-height box with support on $|\omega| \leq 500\pi$.




That the system is *band-limited*, i.e. has a frequency response that is zero outside a specific region, will be crucial in the next part.

(c) For each signal $y(t)$ below, determine whether or not there exists an input $x(t)$ to the system $H$ that could produce $y(t)$ as an output. You do **not** need to determine what inputs could produce these outputs, where possible, just whether or not at least one such input exists.

[2] Extension of Problem 1e of EE 120 Fall 2007 Midterm 1.

We know from the convolution property that

$$Y(j\omega) = H(j\omega)X(j\omega)$$

which means, in particular, that if there is some frequency $\omega_{\text{kill}}$ such that $H(j\omega_{\text{kill}}) = 0$, then

$$Y(j\omega_{\text{kill}}) = H(j\omega_{\text{kill}})X(j\omega_{\text{kill}}) = 0$$

This property is often stated more simply as "LTI systems cannot create new frequencies." Since our system will discard any frequency $\omega$ such that $|\omega| > 500\pi$, we can easily check if a signal is a possible output or not based on whether or not its spectrum is limited to $[-500\pi, 500\pi]$ or not.

(i) $y(t) = \delta(t)$.
No. $Y(j\omega) = 1$, which extends beyond the system bandwidth.

(ii) $y(t) = \cos(100\pi t)$.
Yes. $Y(j\omega) = \pi\delta(\omega + 100\pi) + \pi\delta(\omega - 100\pi)$, which is inside the system bandwidth.

(iii) $y(t) = \cos(375\pi t)$.
Yes. $Y(j\omega) = \pi\delta(\omega + 375\pi) + \pi\delta(\omega - 375\pi)$, which is inside the system bandwidth.

(iv) $y(t) = \sin(600\pi t)$. Would your answer change if the signal was a cosine at the same frequency instead?
No. $Y(j\omega) = \frac{\pi}{j}\delta(\omega - 600\pi) - \frac{\pi}{j}\pi\delta(\omega + 600\pi)$, meaning the output signal has frequency content outside the system bandwidth, which is impossible. The answer would not change if the signal was a cosine at the same frequency instead. A sine and cosine at the same freuqency are only off by a phase shift, which does not change the fact that this signal has spectral content outside the system bandwidth.

(v) $y(t) = \cos(50\pi t)\cos(75\pi t)$.
Yes. The multiplication in time property tells us that $Y(j\omega)$ is the convolution of the two cosines' spectra, scaled by $\frac{1}{2\pi}$. Without explicitly calculating anything, we know this means the spectrum has peaks at $\omega = \pm 125\pi, \pm 25\pi$ since each cosine is composed of two deltas at $\pm 50\pi$ and $\pm 75\pi$.

(vi) $y(t) = \cos(300\pi t)\cos(400\pi t)$.
No. The multiplication in time property tells us that $Y(j\omega)$ is the convolution of the two cosines' spectra, scaled by $\frac{1}{2\pi}$. Without explicitly calculating anything, we know this means the spectrum has peaks at $\omega = \pm 300\pi, \pm 400\pi$ since each cosine is composed of two deltas at $\pm 100\pi$ and $\pm 700\pi$. The $\pm 700\pi$ peaks are illegal, so this signal is not possible as an output of the system $H$.

(vii) $y(t) = 12e^{j300\pi t}$.

Yes. $Y(j\omega) = 12\delta(\omega - 300\pi)$, which is perfectly legal.

(viii) $y(t) = e^{-7t}u(t)$.

No. The spectrum is $Y(j\omega) = \frac{1}{7+j\omega}$, which extends out to $\pm\infty$.

(d) Now, consider the signal

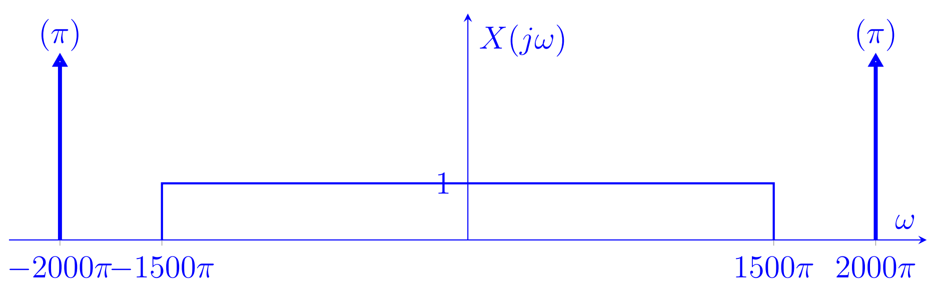
$$x(t) = x_1(t) + x_2(t)$$

where

$$x_1(t) = \cos(2000\pi t) \text{ and } x_2(t) = \frac{\sin(1500\pi t)}{\pi t}$$

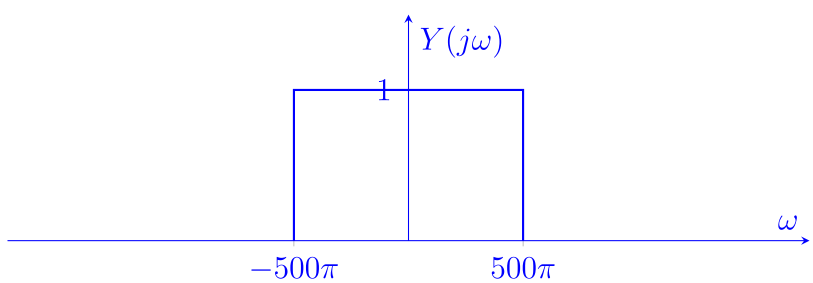
which is input to the LTI system $H$.

(i) Provide a well-labelled plot of $X(j\omega)$, the Fourier Transform of $x(t)$.




(ii) Provide a well-labelled plot of $Y(j\omega)$, the Fourier Transform of $y(t)$, the output of the LTI system for the signal $x(t)$.

The cosine's contribution is completely removed by the system, and the sinc has its bandwidth chopped down to match the system's bandwidth.




(iii) Write an expression for $y(t)$.

Matching to the given transform pair, we have

$$y(t) = \frac{\sin(500\pi t)}{\pi t}$$

at the output.

(iv) Would your answer to the previous part change at all if we had used $x_2(t)$, rather than $x(t)$, as our input? Why or why not?

No, as $x_1(t)$ was completely killed by the system, contributing nothing to the output. This is the EE 120 version of the linear algebraic fact that $A\vec{x} = A(\vec{x} + \vec{v})$ for any vector $\vec{v}$ in the nullspace of a matrix $A$.

(e) Calculate the energy

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$$

of the output signal $y(t)$ you found in the previous part.

From Parseval's theorem, we know

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 d\omega$$

By inspection of our plot in the previous part, the frequency domain energy is $1000\pi$, so we have $E_y = 500$.

**Problem 6: CTFT for Heat Flow** [3] In 1822, Joseph Fourier published *Théorie analytique de la chaleur (The Analytical Theory of Heat)*, in which he developed and solved the heat equation, laying the groundwork for the same Fourier Transform we use today.

Suppose we have a very long rod (one that is essentially infinite in length, so that boundary conditions can be ignored). Let the temperature at position $x \in \mathbb{R}$ along the rod at time $t \geq 0$ be given by $u(x, t)$. The heat equation says that the temperature changes according to

$$\frac{\partial}{\partial t} u(x, t) = \alpha \frac{\partial^2}{\partial x^2} u(x, t) \tag{1}$$

where $\alpha > 0$ is a constant that depends on the thermal diffusivity of the rod's material. As an initial condition, the temperature along any point of the rod at time zero is given as $u(x, 0) = g(x)$.

In this problem, we will use the Fourier Transform to solve the heat equation. However, we will be taking the Fourier Transform with respect to the spatial variable $x$ rather than the time variable $t$ as we typically do. To this end, define

$$U(j\omega, t) = \int_{-\infty}^{\infty} u(x, t) e^{-j\omega x} dx \tag{2}$$

so that

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega, t) e^{j\omega x} d\omega \tag{3}$$

Note that instead of working back and forth between the time and temporal frequency domains as we usually do, we're now working between the space and spatial frequency domains. Fortunately, all the same principles and properties you're familiar with carry over.

[3] Adapted from Problem 5 of EE 120 Spring 2016 Midterm 1.

(a) Show that $\frac{\partial}{\partial t}U(j\omega, t) = -\alpha\omega^2 U(j\omega, t)$.

*Hint:* Think of $\omega$ as some fixed, real constant for this part. Start by plugging $(2)$ into the left hand side of the expression you're asked to show, and then use the heat equation combined with the time (here, space) differentiation property.

Taking the hint, we have

$$\begin{aligned}\frac{\partial}{\partial t}U(j\omega, t) &= \frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} u(x,t)e^{-j\omega x}dx\right]\\ &= \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial t}u(x,t)\right]e^{-j\omega x}dx\\ &= \int_{-\infty}^{\infty}\left[\alpha\frac{\partial^2}{\partial x^2}u(x,t)\right]e^{-j\omega x}dx\\ &= \alpha\int_{-\infty}^{\infty}\left[\frac{\partial^2}{\partial x^2}u(x,t)\right]e^{-j\omega x}dx\\ &= \alpha(j\omega)^2U(j\omega,t)\\ &= -\alpha\omega U(j\omega,t)\end{aligned}$$

as desired.

(b) Now, show that

$$\frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}U(j\omega,t)\right] = e^{\alpha\omega^2 t}\left(\frac{\partial}{\partial t}U(j\omega,t) + \alpha\omega^2 U(j\omega,t)\right) \tag{4}$$

This follows immediately from the product rule for derivatives. For sake of completeness:

$$\begin{aligned}\frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}U(j\omega,t)\right] &= \frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}\right]U(j\omega,t) + e^{\alpha\omega^2 t}\frac{\partial}{\partial t}U(j\omega,t)\\ &= \alpha\omega^2 e^{\alpha\omega^2 t}U(j\omega,t) + e^{\alpha\omega^2 t}\frac{\partial}{\partial t}U(j\omega,t)\\ &= e^{\alpha\omega^2 t}\left(\frac{\partial}{\partial t}U(j\omega,t) + \alpha\omega^2 U(j\omega,t)\right)\end{aligned}$$

(c) Use the results of the previous two parts to show

$$\frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}U(j\omega,t)\right] = 0 \tag{5}$$

Rearranging the result from (a), we obtain

$$\frac{\partial}{\partial t}U(j\omega,t) + \alpha\omega^2 U(j\omega,t) = 0$$

Applying the result of part (b), we obtain

$$\frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}U(j\omega,t)\right] = 0$$

(d) Integrate both sides of the result of the previous part with respect to $t$, and then apply the initial condition given to show that

$$U(j\omega,t) = G(j\omega)e^{-\alpha\omega^2 t} \tag{6}$$

where $G(j\omega)$ is the spatial Fourier Transform of $g(x) = u(x,0)$.

Integrating the result of (c) with respect to $t$ gives:

$$e^{\alpha\omega^2 t}U(j\omega,t) = C(j\omega)$$

where $c(x)$ is some constant with respect to $t$ (i.e., a function purely of $x$) with spatial Fourier Transform $C(j\omega)$ that we still must determine. We can rearrange terms to obtain

$$U(j\omega,t) = e^{-\alpha\omega^2 t}C(j\omega)$$

Finally, the initial condition $u(x,0) = g(x)$ tells us that $U(j\omega,0) = G(j\omega)$, which means $C(j\omega) = G(j\omega)$, giving the result.

(e) Express $u(x,t)$ as a convolution integral in terms of the signal $g$. You may find the Fourier Transform pair

$$\frac{1}{2\sqrt{\pi\kappa}}e^{-x^2/4\kappa} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\kappa\omega^2}$$

to be useful. Here, $\kappa > 0$ is some real constant, and $x$ is the variable over which the transform is taken.

We know from the previous part that

$$U(j\omega,t) = e^{-\alpha\omega^2 t}G(j\omega)$$

By the convolution property of the Fourier Transform, we know $u(x,t)$ is the spatial convolution of $g(x)$ with the inverse Fourier Transform of $e^{-\alpha\omega^2 t}$. For notational convenience, we'll call this spectrum $K(j\omega,t)$ so that the time domain signal is $k(x,t)$. Using the given transform pair, we obtain

$$k(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} e^{-x^2/4\alpha t}$$

and so we can solve for $u(x,t)$ by computing the spatial convolution integral

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} g(x-\chi)k(\chi,t)d\chi \\ &= \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} g(x-\chi)e^{-\chi^2/4\alpha t} d\chi \end{aligned}$$

which is the convolution of $g$ with a Gaussian kernel. Gaussians: they're everywhere.

(f) Given the initial condition $u(x,0) = \delta(x)$, solve for $u(x,t)$ using your result in the previous part.[4]

With $g(x) = \delta(x)$, the convolution greatly simplifies thanks to the sifting property. The delta will pick out only one point in space, at $\chi = x$, giving:

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} \delta(x-\chi)e^{-\chi^2/4\alpha t} d\chi \\ &= \frac{1}{\sqrt{4\pi\alpha t}} e^{-x^2/4\alpha t} \end{aligned}$$

The work above is all that the question required. However, there are several interesting observations we are now equipped to make about how an idealized point source of heat will diffuse through **any** sufficiently long rod, regardless of the material it is made of:

- **At all times, the spatial distribution of heat is Gaussian.** As we increase $t$, the distribution gets wider and wider, corresponding to the heat diffusing through the rod. The *rate* at which an increase in $t$ causes the distribution to spread out is determined by the thermal diffusivity constant $\alpha$. For a given change in time, a larger $\alpha$ will cause the distribution to spread out more, implying that the heat diffuses more quickly.

[4]If you like, you can think of the integral of the rod's temperature over all space, $\int_{-\infty}^{\infty} u(x,t)dx$, as representing the total amount of heat in it at time $t$. This isn't quite correct due to units not working out, but it's a passable analogy for this problem. Using this metaphor, the initial condition $g(x) = \delta(x)$ means that we have one unit of heat placed in the middle of the rod at time $t = 0$, and we want to see how it spreads out.

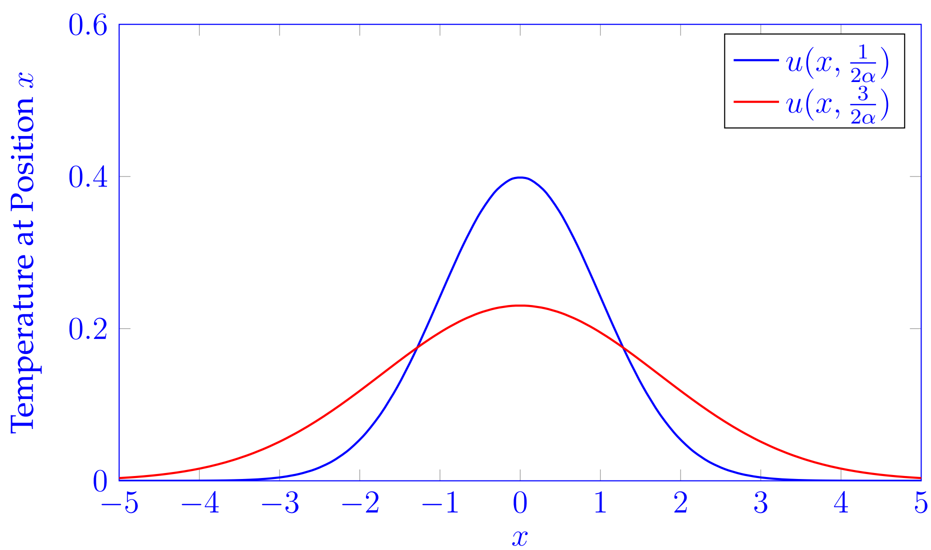
- **Conservation of energy.** The model we used for heat diffusion, given by the heat equation, assumes that all the heat stays in the rod, with none leaving it as it disperses. Consequently, we can see that for all time, the integral of $u(x,t)$ across space is 1. The normalization may not be obvious if you aren't familiar with the Gaussian function, from, for example, a probability class, but you can verify this claim using a computer algebra system. The principle of conservation of energy from physics is in action here — the rod doesn't interact with its environment in our model, so all the heat stays in it.

(g) Determine an expression for, and provide well-labelled plots of, $u(x, t = \frac{1}{2\alpha})$ and $u(x, t = \frac{3}{2\alpha})$ on the same graph. What happens to the spatial distribution of the heat in the rod as time goes on?

We can compute the spatial heat distribution at the specified times as

$$u(x, t = \tfrac{1}{2\alpha}) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
$$u(x, t = \tfrac{3}{2\alpha}) = \frac{1}{\sqrt{6\pi}} e^{-x^2/6}$$

which you may recognize (although this is certainly not necessary) are zero-mean Gaussians with variance 1 and 3, respectively, and shown below.




As time goes on, the heat spreads out through the rod more and more. In the limit as $t \to \infty$, the spatial distribution of heat approaches one that is identically zero, meaning all the heat has dispersed. In practice, the rod will have some finite length $L$, and the distribution will instead approach a uniform one with height $\frac{1}{L}$.